W6/2 Minimal Primes, Artinian Rings

Montag, 24. Februar 2025 08:15

⚠ M noeth $\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec}(A)$: $M_\mathfrak{p}$ noetherian, BUT „$\Leftarrow$" FAILS (e.g. $\prod_{i=1}^{\infty} k$, k field)

<u>Thm 4.7</u> Suppose A is a ring, $a_1, \dots, a_n$ s.t. $(a_1, \dots, a_n) = A$.
If each localization A_{a_i} is noetherian, A is noetherian.

<u>Proof:</u> Observe first: $\forall m_1, \dots, m_n \geq 1$: $(a_1^{m_1}, \dots, a_n^{m_n}) = A$

$\left[\text{Otherwise } \exists M \in \operatorname{Max}(A): (a_1^{m_1}, \dots, a_n^{m_n}) \subseteq M, \text{ but then } (a_1, \dots, a_n) \subseteq \sqrt{(a_1^{m_1}, \dots, a_n^{m_n})} \overset{C2.4}{\subseteq} M \text{ ⚡}\right]$

Consider the localizations $j_i : A \longrightarrow A_{a_i}$.

<u>Claim:</u> $\forall I \trianglelefteq A$: $I = \bigcap_{i=1}^{n} j_i^{-1}(I A_{a_i})$

<u>Proof of Claim:</u> „$\subseteq$" ✓, „$\supseteq$" Let $x \in \bigcap_{i=1}^{n} j_i^{-1}(I A_{a_i})$

$\Rightarrow \forall i$: $j_i(x) = \dfrac{b_i}{a_i^{m_i}}$ with $b_i \in I$, $m_i \geq 0$

$\Rightarrow a_i^{k_i} a_i^{m_i} x = \underbrace{a_i^{k_i} b_i}_{\in I}$ with $k_i \geq 0$.

$\Rightarrow a_i^{k_i + m_i} x \in I \quad \forall i$

Now $(a_1^{k_1+m_1}, \dots, a_n^{k_n+m_n}) = A$

$\Rightarrow \exists c_1, \dots, c_n \in A$: $1 = \sum_{i=1}^{n} c_i a_i^{k_i+m_i} \Rightarrow x = \sum_{i=1}^{n} c_i \underbrace{a_i^{k_i+m_i} x}_{\in I} \in I$.

□ (Claim).

If $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals, then

$\forall i$: $I_1 A_{a_i} \subseteq I_2 A_{a_i} \subseteq \dots$ becomes stationary

$\overset{\text{Claim}}{\Longrightarrow} I_1 \subseteq I_2 \subseteq \dots$ becomes stationary. □

<u>Prop 4.8</u> If $P \in \operatorname{Spec}(A)$, then P contains a minimal prime ideal.
In particular: Every nonzero ring has minimal primes.

<u>Exm:</u> If A is a domain, $\underline{0}$ is the unique minimal prime.
In $\mathbb{Z}/6\mathbb{Z}$ there are two: $(\bar{2})$, $(\bar{3})$.

Proof: Let $\Omega = \{Q \in \operatorname{Spec}(A) : Q \subseteq P\}$. To apply Zorn's Lemma (wrt $\supseteq$)

Proof: Let $\Omega = \{Q \in \mathrm{Spec}(A) : Q \subseteq P\}$. To apply Zorn's Lemma (vrl 2) suffices: If $\Omega_0 \subseteq \Omega$ is a <u>descending</u> chain, then $I := \bigcap_{Q \in \Omega_0} Q$ is prime.

Suppose $a, b \in A$: $ab \in I$. Suppose $a \notin I \Rightarrow \exists Q' \in \Omega_0 : a \notin Q'$

$\Rightarrow \forall Q'' \in \Omega_0 : \quad Q'' \subseteq Q' \Rightarrow a \notin Q'' \Rightarrow b \in Q''$

Hence $b \in Q''$ for <u>all</u> $Q'' \in \Omega_0$. $\square$

The minimal primes of $I \trianglelefteq A$ are the $P \in \mathrm{Spec}(A)$ that are minimal over I (in bij. with min. primes of A/I).

<u>Prop 4.9</u> If A is noetherian, $I \trianglelefteq A$, then I has finitely many min. primes.

„noetherian induction"

<u>Proof</u>: Suppose not. Let $\Omega = \{I \trianglelefteq A : I \text{ has infinitely many min. primes}\}$.

Since $\Omega \neq \emptyset$ and A is noetherian, Ω has a max. element I.

Replacing A by A/I, wlog $I = 0$.

0 cannot be prime $\Rightarrow \exists a, b \in A \setminus \{0\} : \quad ab = 0$.

By maximal choice of I, (a) and (b) have fin. many min. primes

But if P is a min. prime of 0, then $a \in P$ or $b \in P$, so P is a min. prime of (a) or of (b) ↯ $\square$

<u>Cor 4.10</u> Let A be noetherian, $I \trianglelefteq A$

(1) $\sqrt{I} = P_1 \cap \dots \cap P_n$ for some $n \geq 0$, $P_i \in \mathrm{Spec}(A)$

(2) $\exists n \geq 1 : \quad (\sqrt{I})^n \subseteq I$. <u>In particular</u>: $\exists k \geq 1 : \mathcal{N}(A)^k = 0$.

<u>Proof</u>: (1) $\sqrt{I} \overset{C2.4}{=} \bigcap_{\substack{P \in \mathrm{Spec}(A) \\ I \subseteq P}} P = \bigcap_{\substack{P \text{ min. prime} \\ \text{of } I}} P$, and this is a finite intersection by P4.9

(2) Let $\sqrt{I} = (a_1, \dots, a_n)$ with $a_i \in A$.

Take $k \geq 1$ s.t. $a_i^k \in I \quad \forall 1 \leq i \leq n$.

Let $m := kn$.

Any product $x = a_{i_1} \cdots a_{i_m}$ with $i_1, \dots, i_m \in \{1, \dots, n\}$ contains some a_i

Any product $x = a_{i_1} \cdots a_{i_m}$ with $i_1, \dots, i_m \in \{1, \dots, n\}$ contains some a_i at least k times (pigeonhole principle) $\Rightarrow x \in I \Rightarrow (\sqrt{I})^m \subseteq I$. $\square$

4.1 Artinian Rings

<u>Examples:</u> finite rings (e.g. $\mathbb{Z}/n\mathbb{Z}$, $n \neq 0$), $K[x]/(x^n)$ (K field), R/aR R PID, $a \neq 0$; A finite-dimensional K-algebra (K field)

<u>Lemma 4.11</u> A artinian $\Rightarrow$ $\mathrm{Spec}(A) = \mathrm{Max}(A)$

<u>Proof:</u> If $P \in \mathrm{Spec}(A)$, then A/P is an artinian domain, hence a field. $\square$

<u>Prop 4.12</u> Let A be artinian

(1) $\mathrm{Max}(A)$ is finite

(2) $\exists k \geq 1$: $\mathcal{N}(A)^k = 0$

<u>Proof</u> (1) Suppose not, let $M_1, M_2, \dots$ be inf. many distinct max. ideals

$\Rightarrow M_1 \supseteq M_1 \cap M_2 \supseteq M_1 \cap M_2 \cap M_3 \supseteq \dots$ stabilizes

$$\Rightarrow \bigcap_{i=1}^{n} M_i = \bigcap_{i=1}^{n+1} M_i \text{ for some } n \Rightarrow M_{n+1} \supseteq \bigcap_{i=1}^{n} M_i$$

$\xRightarrow[L1.2(2)]{M_{n+1} \text{ prime}} \exists\, 1 \leq i \leq n$: $M_i \subseteq M_{n+1} \Rightarrow M_i = M_{n+1}$ $\lightning$.

(2) Since $\mathcal{N}(A) \supseteq \mathcal{N}(A)^2 \supseteq \mathcal{N}(A)^3 \supseteq \dots$

$\exists k \geq 1$: $\mathcal{N}(A)^k = \mathcal{N}(A)^{k+1}$

<u>Claim:</u> $\mathcal{N}(A)^k = \underline{0}$.

Suppose $\mathcal{N}(A)^k \neq 0$. $\Omega := \{ I \trianglelefteq A : I \cdot \mathcal{N}(A)^k \neq 0 \}$ $\quad (A \in \Omega)$

A artinian $\Rightarrow \exists$ minimal $I \in \Omega$. $\Rightarrow \exists x \in I$: $x \mathcal{N}(A)^k \neq 0$, so in fact $I = (x)$ by minimality.

$\Rightarrow (x \mathcal{N}(A)) \mathcal{N}(A)^k = x \mathcal{N}(A)^{k+1} = x \mathcal{N}(A)^k \neq 0$

$\Rightarrow x \mathcal{N}(A) \in \Omega \xRightarrow[\text{min.}]{} (x) = x \mathcal{N}(A) \Rightarrow x = xn$ with $n \in \mathcal{N}(A)$

$\Rightarrow x\mathcal{N}(A) \in \Omega \underset{\text{min.}}{\Rightarrow} (x) = x\mathcal{N}(A) \Rightarrow x = xn$ with $n \in \mathcal{N}(A)$

But n is nilpotent, so $\exists \ell$: $n^\ell = 0$

$\Rightarrow x = xn = xn^2 = \dots = xn^\ell = 0$ $\lightning$ $x\mathcal{N}(A)^k \neq 0$ $\square$